Vortex dynamics in rotating counterflow and plane Couette and Poiseuille turbulence in superfluid Helium

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Key words: superfluid turbulence; vortices
PACS number(s): 67.25.dk, 47.37,+q

Abstract

An equation previously proposed to describe the evolution of vortex line density in rotating counterflow turbulent tangles in superfluid helium is generalized to incorporate nonvanishing barycentric velocity and velocity gradients. Our generalization is compared with an analogous approach proposed by Lipniacki, and with experimental results by Swanson *et al.* in rotating counterflow, and it is used to evaluate the vortex density in plane Couette and Poiseuille flows of superfluid helium.

1 Introduction

Many researches of quantum vortices in superfluids have been carried out on rotating systems and counterflow situations, both of them with vanishing barycentric velocity gradient [1]–[3]. Evolution equations have been proposed to describe the influence of heat flux and of angular velocity on the vortex dynamics [4] generalizing the well-known Vinen's equation for non-rotating systems [1]–[3, 5]. An interesting challenge is to generalize these vortex evolution equations to include the influence of barycentric flow, which has much practical interest, for instance, in cryogenic applications. Here, we carry out such a generalization and we examine a recent proposal by Lipniacki [6], which opens an interesting perspective but which, on the other side, discloses some aspects which have not been yet settled out with enough clarity.

The aim of this paper is to generalize a previous equation proposed for rotating counterflow superfluid turbulence [4] by emphasizing more explicitly the dynamical role of the rotational of the superfluid velocity $\mathbf{v}_s$, related to quantized vortices. This allows us to write a proposal for the evolution equations of vortices in plane Couette and Poiseuille flows. In Section 3 we review some aspects of rotating counterflow and compare our generalized expression with Lipniacki's proposal [6], which underlines the role of the polarization rather than of rot $\mathbf{v}_s$ itself, and we stress some open problems. In Section 4 we use a thermodynamic formalism to

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relate the dynamical equation for vortices with a new term appearing in the mutual friction force, which we use for a comparison with that one by Lipniacki. In Section 5 we discuss several aspects of Couette and Poiseuille flows of superfluid helium including the presence of quantized vortices.

2 Rotational of superfluid velocity and the dynamics of vortex line density

An evolution equation for the dynamics of quantum vortices in rotating helium under counterflow was proposed in [4], describing the influence of the heat flow and of angular velocity on the vortex line density. In particular, the vortex-line density L was assumed to obey the following equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\beta\kappa L^2 + \left[\alpha_1 V + \beta_2\sqrt{\kappa\Omega}\right] L^{3/2} - \left[\beta_1\Omega + \beta_4 V\sqrt{\frac{\Omega}{\kappa}}\right] L, \tag{2.1}$$

where β, α_1, β_2, β_1, and β_4 are dimensionless coefficients, $\kappa = h/m$ is the quantum of vorticity (m the mass of the ^{4}He atom and h Planck's constant), $V = |\mathbf{V}|$ (with $\mathbf{V} = \mathbf{v}_n - \mathbf{v}_s$) is the counterflow velocity, the relative velocity between averaged normal and superfluid velocities, which is proportional to the heat flux across the system, and $\Omega = |\mathbf{\Omega}|$ is the angular velocity of the container. The values of the coefficients were obtained in [4] by comparison with experimental data of [7] and they were seen to satisfy the relations $\beta_4 = \sqrt{2}\alpha_1$ and $\beta_1 = \sqrt{2}\beta_2 - 2\beta$, which are required on relatively general arguments about the form of solutions. The values of the coefficients appearing in (2.1) were independently calculated in [8], and agree with those obtained in [4]. When $\Omega = 0$, equation (2.1) reduces to the well-known Vinen's equation [5], with parameters α_1 and β being respectively related to the production and destruction of vortices per unit volume and time.

In [4, 9] it was shown that the value of coefficient α_1 depends on the angle between the counterflow velocity $\mathbf{V}$ and Schwarz's binormal vector $\mathbf{I}$ [10] (see equation (3.6)). As observed in [4], $\alpha_1 = \alpha_V \mathbf{I}\cdot\hat{\mathbf{V}}$, with α_V the coefficient appearing in Vinen's equation (pure counterflow) [1]. Schwarz derived Vinen's equation using the vortex filament model obtaining $\alpha_V = \alpha c_1$, where α is the well-known coefficient appearing in the expression of the mutual friction force between vortex lines and the normal fluid and c_1 denotes the average curvature of the tangle (see equation (3.7)). In [4] in the regime of high rotation, the value $\mathbf{I}\cdot\hat{\mathbf{V}} = 1/2$ was found, so indicating that the vortex tangle is highly polarized. Coefficient β is linked to the average squared curvature of the vortices as $\beta\kappa = \alpha\tilde{\beta}c_2^2$, with c_2^2 defined in equation (3.7) and $\tilde{\beta}$ the vortex tension parameter, defined as $\epsilon_V = \kappa\rho_s\tilde{\beta}$ with ϵ_V the energy per unit length of vortex line [1].

These equations lack an important source of vorticity, namely a barycentric velocity gradient, which is known to produce turbulence in many actual flows. Thus, it would be useful to generalize (2.1) by incorporating in it barycentric velocity gradients. A possible way to do so would be simply adding new terms basing on dimensional analysis and on comparison with the observed phenomenology. Instead of proceeding in this way, we will interpret (2.1) in some deeper terms, which will be useful for a consistent incorporation of the velocity gradient.

To generalize equation (2.1) we note that in the particular case of pure rotation Ω is related to rot $\mathbf{v}_s$ as $2\Omega = |\text{rot } \mathbf{v}_s|$, where $\mathbf{v}_s$ is the macroscopic superfluid velocity. As Lipniacki noted

in a different proposal [6], writing an equation such as (2.1) in terms of rot $\mathbf{v}_s$ and V rather than in terms of Ω and V would be more general, because it would reduce to (2.1) for rotation, and it could be applied to other flows as plane Couette or Poiseuille flows (see Section 5), where $|\text{rot } \mathbf{v}_s| = dv_{sx}(z)/dz$, x being the direction of the fluid motion, z the direction orthogonal to the parallel plates, and $v_{sx}(z)$ the macroscopical superfluid velocity, depending only on z.

In this way, the natural generalization of (2.1) would be to rewrite it in terms of rot $\mathbf{v}_s$ as

$$\frac{dL}{dt} = -\beta\kappa L^2 + \left[\alpha_1 V + \frac{\beta_2}{\sqrt{2}}\sqrt{\kappa|\text{rot } \mathbf{v}_s|}\right] L^{3/2} - \left[\frac{\beta_1}{2}|\text{rot } \mathbf{v}_s| + \frac{\beta_4}{\sqrt{2}} V \sqrt{\frac{|\text{rot } \mathbf{v}_s|}{\kappa}}\right] L. \tag{2.2}$$

Equation (2.2) reduces to (2.1) for pure rotation. Besides that, expression (2.2) generalizes (2.1) also on dynamical grounds. Note, indeed, that in (2.1) it is assumed that $|\text{rot } \mathbf{v}_s|$ is equal to 2Ω. However, it will take some time for $\mathbf{v}_s$ to get these values, by starting after some arbitrary initial state. Thus, whereas Ω in (2.1) is taken as an externally fixed parameter, in (2.2) rot $\mathbf{v}_s$ is a dynamical quantity, which must be described by a suitable evolution equation. Then, the form (2.1) will be useful after some transient interval, whereas (2.2) is expected to be valid also for fast changes in $\mathbf{v}_s$. Further, equation (2.2) can be applied also in different situations, as plane Couette and Poiseuille flows. Thus, equation (2.2) is the central point of this paper, as it generalizes (2.1) both to a wider set of external conditions and to a wider domain of dynamical variations.

Comparison with a similar approach by Lipniacki [6] will be useful for a better understanding of both approaches. Lipniacki [6] has essentially proposed to use as variable the so-called "polarity vector" (see also [11]), an important quantity in vortex dynamics, which he linked to the rotational of the averaged superfluid velocity

$$\mathbf{p} = < \mathbf{s}' > = \frac{\int \mathbf{s}' d\xi}{\int d\xi} = \frac{\nabla \times \mathbf{v}_s}{\kappa L}. \tag{2.3}$$

For pure rotation one has $\mathbf{p} = \hat{\mathbf{\Omega}}$ and $\nabla \times \mathbf{v}_s = 2\mathbf{\Omega}$; therefore, we may rewrite equation (2.1) in terms of $\mathbf{p}$. Note that $|\mathbf{p}| \in [0, 1]$ measures the directional anisotropy of the tangent to the vortex lines: in particular $|\mathbf{p}| = 1$ for a system of parallel vortices and $|\mathbf{p}| = 0$ for isotropic tangles. Thus, it is possible to express (2.2) in terms of $\mathbf{p}$ and to group the terms in it in a slightly different way, namely, in two groups, one of them with the factor $VL^{3/2}$ and the other one with kL^2, mimicking in some way the form of the original Vinen's equation. In this way, we rewrite (2.2) as

$$\frac{dL}{dt} = -\beta\kappa L^2 \left[1 - \frac{\beta_2}{\beta}\sqrt{\frac{\Omega}{\kappa L}} + \frac{\beta_1}{\beta}\frac{\Omega}{\kappa L}\right] + \alpha_1 V L^{3/2}\left[1 - \frac{\beta_4}{\alpha_1}\sqrt{\frac{\Omega}{\kappa L}}\right], \tag{2.4}$$

which, recalling $\beta_1 = \sqrt{2}\beta_2 - 2\beta$ and the previously mentioned relation $2\Omega = |\text{rot } \mathbf{v}_s|$ implying $2\Omega/\kappa L = |\mathbf{p}|$, (2.4) assumes the more compact form

$$\frac{dL}{dt} = \alpha_1 V L^{3/2}\left[1 - A\sqrt{|\mathbf{p}|}\right] - \beta\kappa L^2 \left[1 - \sqrt{|\mathbf{p}|}\right]\left[1 - B\sqrt{|\mathbf{p}|}\right], \tag{2.5}$$

where $B = \frac{\beta_1}{2\beta}$ and $A = \frac{\beta_4}{\sqrt{2}\alpha_1}$. In [4], coefficient B was found to be 0.89 while coefficient A is not properly a constant but undergoes a small step from 1 to 1.004 at the first counterflow critical velocity V_{c1}. In this work, as already pointed out, we neglect this step assuming $A = 1$.

When inhomogeneities in the line density L are taken into account, the evolution equation for line density L must include a vortex density flux $\mathbf{J}^L$ [12]

$$\frac{\partial L}{\partial t} + \nabla \cdot \mathbf{J}^L = \sigma^L, \tag{2.6}$$

where σ_L stands for the production term given by the right-hand side of equation (2.5). The form of $\mathbf{J}^L$ contains a convective contribution, $L\mathbf{v_L}$ with $\mathbf{v_L}$ the velocity of vortex lines with respect to the laboratory frame, and a diffusive contribution. In some situations, when the rate of variation of the perturbations is higher than the reciprocal of the relaxation time of the diffusive flux [12, 13], one must take $\mathbf{J}^L$ as an independent variable [14]. Here, neglecting the relaxation time of $\mathbf{J}^L$ and considering isothermal situations, we take for $\mathbf{J}^L$ the following simple law, where the diffusive contribution is analogous to Fick's diffusion law

$$\mathbf{J}^L = -\tilde{D}\nabla L + L\mathbf{v_L}. \tag{2.7}$$

The coefficient $\tilde{D}$ (of the order of κ [12],[13]) is the diffusion coefficient of vortex lines.

For a general hydrodynamic description, the evolution equations for $\mathbf{v}_n$ and $\mathbf{v}_s$ are needed. In particular, the evolution of $\mathbf{v}_s$ is necessary to describe the evolution of rot $\mathbf{v}_s$ in equation (2.2). A set of equations frequently used are the Hall-Vinen-Bekarevich- Khalatnikov equations [1, 15], which in an inertial frame are written as

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \rho_n(\mathbf{v}_n \cdot \nabla)\mathbf{v}_n = -\frac{\rho_n}{\rho}\nabla p_n - \rho_s S \nabla T + \mathbf{F}_{ns} + \eta \nabla^2 \mathbf{v}_n, \tag{2.8}$$

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \rho_s(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = -\frac{\rho_s}{\rho}\nabla p_s + \rho_s S \nabla T - \mathbf{F}_{ns} + \rho_s \mathbf{T}. \tag{2.9}$$

Here, p_n and p_s are effective pressures, defined as $\nabla p_n = \nabla p + (\rho_s/2)\nabla V^2$, $\nabla p_s = \nabla p - (\rho_n/2)\nabla V^2$, p the total pressure, S the entropy, η the dynamic viscosity of the normal component, and $\rho_s \mathbf{T}$ the vortex tension force, which vanishes for rectilinear vortices and for isotropic vortex tangles, but which may be relevant in other situations. In the situations considered in this paper, we will have $\mathbf{T} = \mathbf{0}$.

To describe the motion we need an expression for $\mathbf{F}_{ns}$, the mutual force between normal and superfluid components. The usual expression by Hall, Vinen, Bekarevich and Khalatnikov is [1]

$$\mathbf{F}_{ns} = \alpha \rho_s \kappa L \left[\hat{\mathbf{p}} \times [\mathbf{p} \times (\mathbf{V} - \mathbf{v_i})] + \frac{\alpha'}{\alpha} \hat{\mathbf{p}} \times (\mathbf{V} - \mathbf{v_i}) \right], \tag{2.10}$$

with α and α' being friction coefficient depending on temperature, and $\mathbf{v_i}$ the "self-induced velocity", which in the HVBK equations is approximated by

$$\mathbf{v_i} = \tilde{\beta} \nabla \times \hat{\mathbf{p}}. \tag{2.11}$$

The expression for $\mathbf{F}_{ns}$ must be consistent with the dynamics of L. In Section 4 we will explore how (2.10) should be modified in order to be consistent with the evolution equation (2.5), and in Section 5 we will combine the equations in an analysis of plane Couette and Poiseuille flows in steady conditions.

3 Rotating counterflow

In this Section, we investigate the proposed equation (2.2) for a rotating superfluid helium inside a cylindric container in the absence and in presence of counterflow and we compare some results of our proposal with those of Lipniacki [6], and with the experimental data of Swanson *et al.* [7]. These authors considered a rotating container filled of helium II with an external counterflow $\mathbf{V}$ parallel to the angular velocity $\mathbf{\Omega}$ of the container. For high angular velocities, they observed two critical counterflow velocities V_{c1} and V_c such that for $0 \leq V \leq V_c$ the line density L is approximately independent of V, undergoing only a small step (about 0.4%) at the first critical velocity V_{c1} whereas for $V \geq V_c$ the line density L grows with V. Here we will neglect the small variation of L at the first critical velocity V_{c1}, because our proposal reduces to (2.1) in this situation — which was already carried out in [4] — and it is not necessary for the comparison with Lipniacki's proposal because the latter is valid only for $V \geq V_c$.

3.1 Pure rotation

First of all, we consider the simplest situation of a cylindric container rotating around its axis. It is known that when the angular velocity Ω exceeds a critical value Ω_c and the stationary state is reached, vortex lines parallel to the rotation axis are present whose number density follows the law $L = 2\Omega/\kappa$. The presence of these vortices may be explained observing that when the container begins to rotate the viscous normal fluid rotates with it, whereas the superfluid remains initially at rest, due to its vanishing viscosity. In this situation, the difference between $\mathbf{v}_n$ and $\mathbf{v}_s$ is zero along the rotation axis, but it is maximum near the walls of the container, that is, the counterflow velocity increases for increasing distance from the axis. In this way the remnant vortices, which are formed during the cooling of helium and which are pinned to the walls, are influenced by the counterflow velocity. This implies the growth of these vortices in agreement with the dynamical description proposed by Schwarz. According to this idea, vortices will grow near the walls, due to the relative velocity between normal and superfluid velocity, and will migrate towards the bulk of the system, forming in the stationary situation a regular array of vortices parallel to the rotation axis.

The presence of vortices couples the normal fluid and the superfluid through the mutual friction force so that vortices are dragged by the normal fluid, and the average superfluid velocity $\mathbf{v}_s$ becomes different from zero. This fact justifies the relation $\nabla \times \mathbf{v}_s^{\Omega} = 2\mathbf{\Omega}$ and the substitution of $2\Omega/\kappa L = |\mathbf{p}|$ in equation (2.1). At the light of the new arguments, in the case of pure rotation the vortex line density becomes $L = 2\Omega/\kappa$, which implies $|\mathbf{p}| \equiv 1$.

Consider now equation (2.5) in the case of pure rotation, when the stationary solution is reached, that is $V =< |\mathbf{v}_n - \mathbf{v}_s| >\approx 0$. In this case (2.5) has two stationary solutions, $|\mathbf{p}| = 1$ and $|\mathbf{p}| = 1/B^2$. As one can easily verify, the solution $|\mathbf{p}| = 1$ is stable if $B < 1$ and this is the case because the coefficient B was found to be 0.89 [4]. To describe the non-stationary regime, one needs to introduce equations (2.8) and (2.9) for the averaged normal and superfluid velocities.

3.2 Fast rotation and external counterflow

Equation (2.5) can be also written as

$$\frac{\mathrm{d}L}{\mathrm{d}t} = L^{3/2}\left(1-\sqrt{|\mathbf{p}|}\right)\left[\alpha_1 V - \beta\kappa L^{1/2}\left(1-B\sqrt{|\mathbf{p}|}\right)\right]. \tag{3.1}$$

As we have pointed out above, pure rotation is well described by (3.1), because in this situation $\mathbf{v}_s \equiv \mathbf{v}_s^{\Omega}$ and $|\mathbf{p}| = 1$ is a stationary solution of (3.1), meaning complete polarization. The non-zero stationary solutions of (3.1) are

$$|\mathbf{p}| = 1 \qquad \text{and} \qquad L^{1/2} = \frac{\alpha_1}{\beta\kappa}V + B\sqrt{\frac{|\nabla\times\mathbf{v}_s|}{\kappa}}. \tag{3.2}$$

To study the stability of the solution $|\mathbf{p}| = 1$, we linearize Eq. (3.1) for the perturbations. In the hypothesis that the perturbation δ does not modify the vorticity $\vec{\omega} = \text{rot } \mathbf{v}_s$, the relation $\delta|\mathbf{p}| = -(|\mathbf{p}|/L)\delta L$ is obtained, which allows us to obtain the following evolution equation for the perturbation δL

$$\left(\frac{\partial\delta L}{\partial t}\right)_{|\mathbf{p}|=1} = \left[\frac{\alpha_1 V}{2L^{1/2}} - \frac{1}{2}\beta\kappa(1-B)L\right]\delta L. \tag{3.3}$$

From the previous equation it follows that the solution $|\mathbf{p}| = 1$ is stable for V less than

$$V_c = \frac{\beta}{\alpha_1}(1-B)\sqrt{|\nabla\times\mathbf{v}_s|\kappa}, \tag{3.4}$$

which corresponds to the critical velocity V_c in the experiments of Swanson et al. [7]. Note that if $B = 1$ in Eq. (3.4), the critical counterflow velocity for which the straight vortex lines parallel to the rotation axis become unstable is zero. From an experimental point of view this is not the case because a nonvanishing critical velocity is observed, confirming the value, $B = 0.89 < 1$, obtained in reference [4].

For counterflow velocity higher than the critical velocity (3.4), the solution $|\mathbf{p}| = 1$ becomes unstable, and the line density L assumes the value (3.2b) which depends on V and $|\text{rot } \mathbf{v}_s|$.

Now, we consider the second term in the right hand side of (3.2b), namely $B\sqrt{\frac{|\nabla\times\mathbf{v}_s|}{\kappa}}$. For low values of the counterflow velocity, the vorticity is essentially due to the rotation, and therefore we put $|\nabla\times\mathbf{v}_s| = 2\Omega$, recovering the results obtained in [4].

3.3 Comparison with Lipniacki's proposal

Recently a hydrodynamical model of superfluid turbulence was proposed by Lipniacki [6], mainly with the aim to studying the hydrodynamics of partially polarized tangles arising in rotating counterflow or in plane Couette flow. Thus, it is interesting to compare with his work, whose aims are similar to ours.

Lipniacki writes Vinen's equation as

$$\frac{dL}{dt} = \alpha L^{3/2} c_1(|\mathbf{p}|)\mathbf{I}\cdot\mathbf{V} - \beta\alpha_2 c_2^2(|\mathbf{p}|)L^2, \tag{3.5}$$

where β is a constant of the order of κ, and α the friction coefficient appearing in the expression of the mutual friction force; $\mathbf{I}$ is the binormal vector,

$$\mathbf{I} = \frac{<\mathbf{s}'\times\mathbf{s}''>}{<|\mathbf{s}''|>}, \tag{3.6}$$

defined by Schwarz [10] to describe the polarization of the binormal $\mathbf{s}' \times \mathbf{s}''$, of the vortex lines, with $\mathbf{s}'$ and $\mathbf{s}''$ being the first and second derivatives of the curve $\mathbf{s}(\xi)$ describing a vortex line with respect to the arc-length ξ, $\mathbf{s}'$ the unit tangent along the line and $\mathbf{s}''$ the curvature vector.

The coefficients c_1 and c_2^2 measure the average curvature and curvature squared of the tangle, respectively. They are given, according to the microscopic model by Schwarz [10], by

$$c_1 = \frac{1}{\Lambda L^{3/2}} \int |s''| d\xi, \qquad c_2^2 = \frac{1}{\Lambda L^2} \int |s''|^2 d\xi, \tag{3.7}$$

where Λ is the volume on which one makes the averaging indicated in (3.7). Lipniacki proposes that c_1 and c_2^2 should depend on the polarization $|\mathbf{p}|$, and that they should vanish for completely polarized tangles because in this case $\mathbf{s}'' = 0$ for all the vortex lines. To describe the reduction in c_1 and c_2^2 with respect to its usual variable for a nonpolarized tangle, which will be designed as c_{10} and c_{20}^2, respectively, he assumes that

$$c_1(|\mathbf{p}|) \simeq c_{10} \left[1 - |\mathbf{p}|^2\right], \qquad c_2^2(|\mathbf{p}|) \simeq c_{20}^2 \left[1 - |\mathbf{p}|^2\right]^2. \tag{3.8}$$

In contrast, our expression (2.5) could be interpreted in this perspective as

$$c_1(|\mathbf{p}|) \simeq c_{10} \left[1 - \sqrt{|\mathbf{p}|}\right], \qquad c_2^2(|\mathbf{p}|) \simeq c_{20}^2 \left[1 - \sqrt{|\mathbf{p}|}\right] \left[1 - B\sqrt{|\mathbf{p}|}\right]. \tag{3.9}$$

Therefore, it arises the question of the comparison of both equations (2.5) and (3.5) with the experimental data, and a deeper understanding of the influence of polarity on the coefficients c_1 and c_2^2.

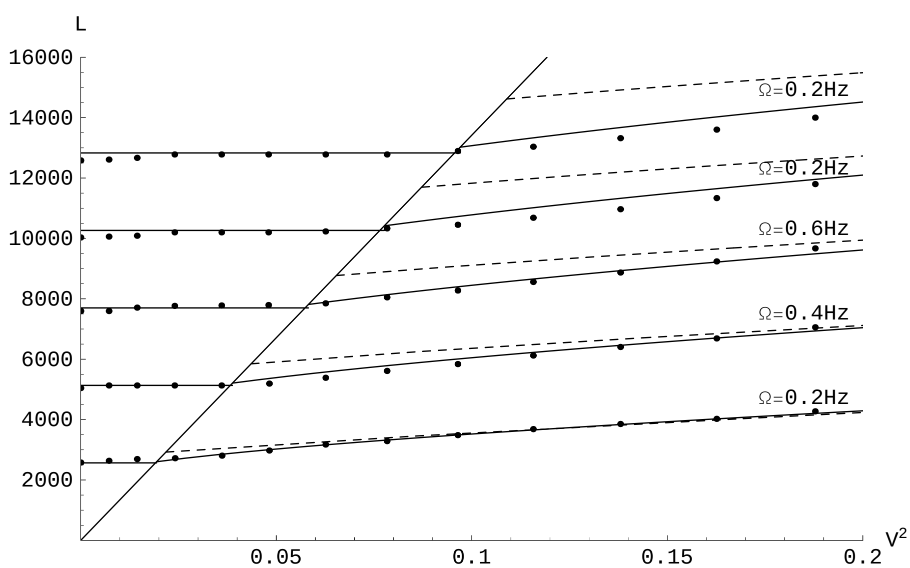

Figure 1: Comparison of the stationary solutions of Lipniacki's model (3.10) (dashed line) and Jou and Mongiovì's model (2.5) (black line) with the experimental data (solid circles) by Swanson *et al.* for counterflow velocity bigger than the second critical velocity V_c and angular velocity 0.2 Hz, 0.4 Hz, 0.6 Hz, 0.8 Hz and 1 Hz. Lipniacki's model does not give the horizontal part of the plot, corresponding to $V < V_c$.

The evolution equation for the vortex line density L, proposed by Lipniacki in [6], has the explicit form

$$\frac{dL}{dt} = \tilde{\alpha} I_0 c_{10} V L^{3/2} \left[1 - |\mathbf{p}|^2\right] - \tilde{\beta} \alpha c_{20}^2 L^2 \left[1 - |\mathbf{p}|^2\right]^2, \tag{3.10}$$

where $I_0 = \mathbf{I} \cdot \hat{\mathbf{V}}$, and the subscript 0 stands for independence of I_0 on V and L. The author chooses for I_0 the same values found in pure counterflow, in such a way to not consider the anisotropy of the vortex tangle, which is present owing of the high values of rotation considered in the experiments by Swanson *et al.* [7].

Equation (3.10), as the author remarks, does not describe any of the two critical velocities, V_{c1} or V_c, of the experiments of Swanson et al. [7]. Lipniacki's aim is instead to describe the relation between angular velocity, counterflow, and line-length density for polarized tangles above the second critical velocity V_c. This implies the need of a comparison, in the uniform steady rotation and counterflow, between (2.5), (3.10) and the experimental data of Swanson et al. [7].

The stationary solutions of the equation (3.10) are $|\mathbf{p}| = 1$ (which however is unstable) and

$$L = \frac{L_H}{\left(1 - (L_\omega/L)^2\right)^2}, \tag{3.11}$$

where

$$L_H = V^2 \left(\frac{c_{10} I_0}{\beta c_{20}^2}\right)^2 \quad \text{and} \quad L_\omega = \frac{|\mathrm{rot}\mathbf{v}_s|}{\kappa} = \frac{2\Omega}{\kappa} \tag{3.12}$$

are the steady state vortex-line density in pure counterflow and in pure rotation, respectively.

In Fig. 1, we compare the results of equations (2.5) and (3.10) with the experimental data of the Fig. 2 of Swanson's experiments. It follows that (2.5) (black line) describes better the experimental data (solid circle) than (3.10) (dashed line), not only for $V > V_c$, but it also yields the horizontal branch of the results for $V < V_c$, which are not described by equation (3.10). Comparison with experimental data shows that in the considered range of values of V and Ω equation (2.5) fits better the experimental results.

A reason for the difference between proposals (2.5) and (3.10) could be related not to the evaluation of the integrals in (3.7) but to a different microscopical interpretation of some terms in the evolution equation for L. Schwarz's derivation [10] is based on the dynamics of vortex breaking and reconnection, and its production and destruction terms tend to zero for completely polarized systems, as rightly pointed out by Lipniacki. However, the origin of the rotational terms in (2.1) could be completely different. It is known that in rotating superfluid helium the vortices grow near the walls due to the rotation, and drift towards the center of the system, where they find a repulsion due to other vortices. These forces are different from zero even for completely polarized vortices, in contrast to the terms from (3.7). It could then be that the vanishing of the terms in (2.5) as $1 - \sqrt{|\mathbf{p}|}$ had a different physical origin than the vanishing proposal by Lipniacki from a different model. These open questions stress the need of the inclusion of rotational effects in a more general version of Schwarz's derivation of Vinen's equation.

4 Thermodynamic analysis of polarized superfluid turbulence

In this Section, we will perform two modifications of the expression of the mutual friction force, as used in the HVBK model, which are necessary to incorporate the anisotropy of the vortex tangle and to insure the thermodynamic consistency of the evolution equation for L and for $\mathbf{v}_s$, according to the formalism of linear irreversible thermodynamics [9, 16]. Since (2.5) differs from the usual Vinen's equation, it is logical to ask how these modifications will

change the form of $\mathbf{F}_{ns}$. For the sake of simplicity, we will neglect here the contribution of the self-induced velocity in (2.10).

First, we will take into account the anisotropy of the tangle introducing the tensor $\mathbf{\Pi} = \mathbf{\Pi}^s + \mathbf{\Pi}^a$, studied in [9], [11],

$$\mathbf{\Pi}^s \equiv \frac{3}{2} < \mathbf{U} - \mathbf{s}'\mathbf{s}' >, \qquad \mathbf{\Pi}^a \equiv \frac{3}{2}\frac{\alpha'}{\alpha} < \mathbf{W} \cdot \mathbf{s}' > . \tag{4.1}$$

In this equation $\mathbf{s}'$ is the unit vector tangent to the vortex lines, $\mathbf{s}'\mathbf{s}'$ is the diadic product, $\mathbf{U}$ is the unit matrix, $\mathbf{W}$ is the Ricci third-order tensor and the angular brackets stands for the average over vortex lines in a given volume. The tensor $\mathbf{\Pi}^s$ describes the orientation of the tangents $\mathbf{s}'$ of the vortex lines, and the tensor $\mathbf{\Pi}^a$ — associated to an axial vector — describes the polarization; in other words, $\mathbf{\Pi}^a$ is related to the first-order moment of the orientational distribution function of $\mathbf{s}'$ and $\mathbf{\Pi}^s$ is related to second-order moment. As shown in Ref. [11], using tensor $\mathbf{\Pi}$, the mutual friction force can be written

$$\mathbf{F}_{ns} = -\alpha\rho_s\kappa L\frac{2}{3}\mathbf{\Pi} \cdot \mathbf{V}. \tag{4.2}$$

If we suppose isotropy in the tangle, it results $\mathbf{\Pi}^s = \mathbf{U}$, $\mathbf{\Pi}^a = \mathbf{0}$ and one finds the usual expression

$$\mathbf{F}_{ns} = -\frac{2}{3}\alpha\rho_s\kappa L\mathbf{V}. \tag{4.3}$$

The tensor $\mathbf{\Pi}$ in (4.2) allows one to deal under a same formalism an array of parallel straight vortices as well as an isotropic tangle, and also the intermediate situations.

Now, we follow the general lines of [9], [17] to propose a modification to (4.2) with the aim to determine an evolution equation for $\mathbf{v}_s$ consistent with (2.5). According to the formalism of nonequilibrium thermodynamics one may obtain evolution equations for $\mathbf{v}_s$ and L by writing $d\mathbf{v}_s/dt$ and dL/dt in terms of their conjugate thermodynamic forces $-\rho_s\mathbf{V}$ and ϵ_V. The evolution equation (2.9) for $\mathbf{v}_s$, neglecting inhomogeneous contributions of pressure, temperature and velocity, in an inertial frame, is written

$$\rho_s\frac{d\mathbf{v}_s}{dt} = -\mathbf{F}_{ns} = \alpha\rho_s\kappa L\frac{2}{3}\mathbf{\Pi} \cdot \mathbf{V}. \tag{4.4}$$

However, in the right-hand side of (4.2) must be included additional contributions to make (4.4) thermodynamically consistent with (2.5).

In a way similar to that presented in [9], we write $d\mathbf{v}_s/dt$ and dL/dt in matrix form in the system (4.5). In it, we write the equation for L in the form given in equation (2.5) and by means of Onsager-Casimir reciprocity we obtain an additional contribution to the evolution equation for $\mathbf{v}_s$. The result is

$$\begin{pmatrix} \frac{d\mathbf{v}_s}{dt} \\ \frac{dL}{dt} \end{pmatrix} = L \begin{pmatrix} -\frac{1}{\rho_s}\alpha\kappa\frac{2}{3}\mathbf{\Pi} & \pm\frac{\alpha_1}{\rho_s}L^{1/2}\left(1-\sqrt{|\mathbf{p}|}\right)\widehat{\mathbf{V}} \\ -\frac{\alpha_1}{\rho_s}L^{1/2}\left(1-\sqrt{|\mathbf{p}|}\right)\widehat{\mathbf{V}} & -\frac{1}{\epsilon_V}L\left(1-\sqrt{|\mathbf{p}|}\right)\left(1-B\sqrt{|\mathbf{p}|}\right) \end{pmatrix} \begin{pmatrix} -\rho_s\mathbf{V} \\ \epsilon_V \end{pmatrix} \tag{4.5}$$

The sign ambiguity present in that equation comes in a natural way from the Onsager-Casimir reciprocity relation. Indeed, in Feynman-Vinen view, L is a scalar quantity which does not change under time reversal, unlike the superfluid velocity $\mathbf{v}_s$ which changes sign. According to Onsager-Casimir, this leads to antisymmetry of crossed coefficients thus leading to the $+$ sign. In Schwarz view, L possesses vectorial properties and it would change on time reversal,

just like the superfluid velocity. This leads to the symmetry of the kinetic coefficients in the matrix in (4.5), i.e. to the - sign in the upper right-hand term. Below, we will directly take the minus sign, for the sake of a more direct comparison with the work by Lipniacki.

Therefore the equation for $d\mathbf{v}_s/dt$ becomes

$$\rho_s \frac{d\mathbf{v}_s}{dt} = \alpha\rho_s\kappa L \frac{2}{3}\mathbf{\Pi}\cdot\mathbf{V} - \epsilon_V \alpha_1 L^{1/2}\left(1-\sqrt{|\mathbf{p}|}\right)\widehat{\mathbf{V}}. \tag{4.6}$$

The new term not contained in the evolution equation (4.4) for $\mathbf{v}_s$ is the coupling term between $d\mathbf{v}_s/dt$ and ϵ_V in the matrix in (4.5). Note that this term depends on the direction but not on the modulus of V. This class of terms are called dry-friction terms.

Observing that in the steady state (L, $|\mathrm{rot}\mathbf{v}_s|$ and $\mathbf{V}$ constant) the solutions of vortex line density equation (2.5) can be written as

$$L^{1/2} = \sqrt{\frac{|\mathrm{rot}\mathbf{v}_s|}{\kappa}}, \qquad \text{for } \ 0 < V < V_c, \tag{4.7}$$

$$L^{1/2} = \frac{\alpha_1}{\beta\kappa}(V - V_c) + \sqrt{\frac{|\mathrm{rot}\mathbf{v}_s|}{\kappa}}, \qquad \text{for } \ V > V_c, \tag{4.8}$$

and substituting them in (4.6), we obtain the following expression for the coupling force

$$\mathbf{F}_{coupl} = -\epsilon_V\alpha_1\left[L^{1/2} - \sqrt{\frac{|\mathrm{rot}\mathbf{v}_s|}{\kappa}}\right]\widehat{\mathbf{V}} = 0, \qquad \text{for } \ V < V_c, \tag{4.9}$$

$$\mathbf{F}_{coupl} = -\epsilon_V\alpha_1\left[L^{1/2} - \sqrt{\frac{|\mathrm{rot}\mathbf{v}_s|}{\kappa}}\right]\widehat{\mathbf{V}} = \epsilon_V\frac{\alpha_1}{\beta\kappa}(V-V_c)\widehat{\mathbf{V}}, \qquad \text{for } \ V > V_c. \tag{4.10}$$

As a consequence, for $V < V_c$ the coupling force is absent (as in pure rotation) while, for $V > V_c$, when the array of rectilinear vortex lines becomes a disordered tangle, the additional term (4.9) appears. Indeed, in a almost-steady state (L and $|\mathrm{rot}\mathbf{v}_s|$ constant), for $V < V_c$, equation (4.4) would be valid, with L expressed by (4.7), whereas, for $V > V_c$ it would become

$$\frac{d\mathbf{v}_s}{dt} = \alpha\kappa L\frac{2}{3}\mathbf{\Pi}\cdot\mathbf{V} + \epsilon_V\frac{\alpha_1}{\beta\kappa\rho_s}(V-V_c)\widehat{\mathbf{V}}, \tag{4.11}$$

with L expressed by (4.8). Summarizing, in steady states for $V < V_c$ the dry-friction force is absent, while it appears for $V > V_c$, when the array of rectilinear vortex lines becomes a disordered tangle. Thus V_c indicates the threshold not only of the vortex line dynamics but also of the friction acting on the velocity $\mathbf{v}_s$ itself; this seems logical, as both variables are mutually related.

Summarizing, in this Section we have proposed to substitute the expression (4.3) of the mutual friction force used in the HVBK model with

$$\mathbf{F}_{ns} = -\alpha\rho_s\kappa L\frac{2}{3}\mathbf{\Pi}\cdot\mathbf{V} - \epsilon_V\alpha_1 L^{3/2}\left(1-\sqrt{|\mathbf{p}|}\right)\widehat{\mathbf{V}} \tag{4.12}$$

for the sake of thermodynamic consistency with (2.5).

To complete the comparison between Lipniacki's and our model, we analyze in both models the expression of the mutual friction force, which in HVBK equation is expressed by (2.10), while in general terms it is expressed as

$$\mathbf{F}_{ns} = \alpha\rho_s\kappa L < \mathbf{s}'\times[\mathbf{s}'\times(\mathbf{V}-\mathbf{v_i})] > + \alpha'\rho_s\kappa L < \mathbf{s}'\times(\mathbf{V}-\mathbf{v_i}) > . \tag{4.13}$$

Lipniacki neglects the coefficient α' in Eq. (4.13), and he approaches the quantity $< \mathbf{s}' \times (\mathbf{s}' \times \mathbf{V}) > \simeq [< \mathbf{s}'\mathbf{s}' > - \mathbf{U}]\mathbf{V} = \mathbf{I}_v - \mathbf{V}$ (where $\mathbf{I}_v =< \mathbf{s}'(\mathbf{s}' \cdot \mathbf{V}) >$) with

$$< \mathbf{s}' \times (\mathbf{s}' \times \mathbf{V}) > \simeq \mathbf{p} \times (\mathbf{p} \times \mathbf{V}) - \frac{2}{3}(1 - |\mathbf{p}|^2)\mathbf{V}, \tag{4.14}$$

and the quantity $< \mathbf{s}' \times (\mathbf{s}' \times \mathbf{v_i}) > \simeq \tilde{\beta} < \mathbf{s}' \times \mathbf{s}'' > = \tilde{\beta} c_1 L^{1/2} \mathbf{I}$ with

$$< \mathbf{s}' \times (\mathbf{s}' \times \mathbf{v_i}) > \simeq -\tilde{\beta} I_0 c_{10} (1 - |\mathbf{p}|^2) L^{1/2} \hat{\mathbf{V}}. \tag{4.15}$$

In explicit terms he uses

$$\mathbf{F}_{ns} = \alpha\kappa\rho_s L \left[\mathbf{p}(\mathbf{p} \cdot \mathbf{V}) - \mathbf{V}\frac{2 + |\mathbf{p}|^2}{3} + \beta I_0 c_{10} (1 - |\mathbf{p}|^2) L^{1/2} \hat{\mathbf{V}} \right]. \tag{4.16}$$

So in the work of Lipniacki, the tensor $\frac{2}{3}\mathbf{\Pi}^s = < \mathbf{U} - \mathbf{s}'\mathbf{s}' >$ assumes the expression:

$$\frac{2}{3}\mathbf{\Pi}^s \simeq [\mathbf{U} - \mathbf{pp}] + \frac{2}{3}(1 - |\mathbf{p}|^2)\mathbf{U} = \frac{5 - 2|\mathbf{p}|^2}{3}\mathbf{U} - \mathbf{pp}. \tag{4.17}$$

Note that (4.17) does not respect the relation $trace[< \mathbf{U} - \mathbf{s}'\mathbf{s}' >] = 2$, following from the normalized character of $\mathbf{s}'$, if $|\mathbf{p}| \neq 1$. In fact it is

$$\text{trace} \left[\frac{5 - 2|\mathbf{p}|^2}{3}\mathbf{U} - \mathbf{pp} \right] = 5 - 3|\mathbf{p}|^2. \tag{4.18}$$

The last term in (4.16) is a consequence of the drift of the tangle in the direction of the counterflow, caused by its anisotropy, where $\mathbf{I} = I_0\hat{\mathbf{V}}$. This term is substituted in our model by the last term in (4.6), which we can rewrite, recalling that $\epsilon_V = \rho_s\kappa\tilde{\beta}$ and $\alpha_1 = \alpha c_{10} I_0$ as

$$\mathbf{F}_{coupl} = -\epsilon_V \alpha_1 L^{3/2} \left(1 - \sqrt{|\mathbf{p}|}\right) \widehat{\mathbf{V}} = -\rho_s\kappa\tilde{\beta}\alpha c_{10} I_0 L^{3/2} \left(1 - \sqrt{|\mathbf{p}|}\right) \widehat{\mathbf{V}}. \tag{4.19}$$

As it is seen, this term differs from the one of Lipniacki, in the contribution due to the polarization of the tangle, which in our approach depends on $1 - \sqrt{|\mathbf{p}|}$, and in Lipniacki's one on $1 - |\mathbf{p}|^2$. We note also that, in this interpretation, we must choose the negative sign in the expression of this coupling term, in agreement with the microscopic derivation of the filament model by Schwarz.

Lipniacki does not consider the tension $\mathbf{T}$. For a comparison with our work, we must observe that in Lipniacki's model the quantity $< \mathbf{s}'\mathbf{s}' >$ is approximated by $\mathbf{pp}$, and this approximation is correct only if most of the vortex lines in the volume have the same direction.

In Ref. [11] we have provided a microscopic paramagnetic analogy to relate $\mathbf{p} =< \mathbf{s}' >$ with $\mathbf{\Omega}$ and $\mathbf{V}$, in the case of simultaneous counterflow and rotation, but we have not studied the statistic of the curvature vector $\mathbf{s}''$. In contrast, Lipniacki leaves open the value of $\mathbf{p}$ and makes some simple hypotheses about $< |\mathbf{s}''| >$ and $< |\mathbf{s}''|^{\mathbf{2}} >$ in his analysis of the possible influence of polarization in the Vinen's equation.

A further difference between our model and that of Lipniacki refers to the form of the vortex flux for which he writes

$$\mathbf{J}^L = L\mathbf{v}^L = L \left[\mathbf{v}_s + \alpha\mathbf{p} \times \mathbf{V} + \beta\alpha I_0 c_{10} (1 - |\mathbf{p}|^2) L^{1/2} \hat{\mathbf{V}} + \beta\alpha \mathbf{I_k} L^{1/2} \right] \tag{4.20}$$

where the vector $\mathbf{I_k}$ represent the curvature of $\vec{\omega}_s$ lines. This last term is exactly zero if the vortex lines are closed, isotropic of straight, and otherwise it is expected to be small, except for the case when all the vortex lines are parallel to each other but bent. This is only the convective contribution, to which it should be added the diffusive contribution $\mathbf{J}^L = -\tilde{D}\nabla L$.

5 Vortex-line density in steady plane flows

In equation (2.2) (and equation (2.5)) we have rewritten previous equation (2.1) for rotating counterflow turbulence in liquid helium in terms of $|\text{rot } \mathbf{v}_s|$. For pure rotation, $|\text{rot } \mathbf{v}_s| = 2\Omega$ and we just have our original equation, but (2.2) may be also used to describe situations with barycentric motion as plane Couette and Poiseuille flows (without external heat flux) between two parallel plates. Here we will consider two plates separated by a distance D, one at rest and the other one moving at velocity $\mathbf{V}_0$ (Couette flow), or plane Poiseuille flow, given by a longitudinal pressure gradient along the direction of two parallel quiescent walls. Here, we will deal with steady states and quasi-stationary states. We will assume that the flow of the normal component remains laminar. This requires that the Reynolds number DV_0/η, with η the viscosity of the normal component and V_0 the characteristic velocity of flow, is sufficiently small. On the other side, in analogy with the rotating container, we assume that the velocity V_0 is sufficiently high to neglect the "effects of the walls" [18]. The essential problem in both cases is to find the distribution of the superfluid velocity, the vortex density and the mutual friction force. We will focus our attention mainly to steady state situations, as simple illustration of the changes implied by the new equations (2.5) and (4.6), for L and $\mathbf{v}_s$.

5.1 Plane Couette flow

We assume two plane surfaces at $z = 0$ and $z = D$ such that the second one moves parallely to the first one at the velocity $\mathbf{V}_0$, and that the relative velocity between normal and superfluid velocities has a profile $\mathbf{V} = (V_x(z), 0, 0)$. The dynamics of vortex formation is similar to that in the rotating cylinder. When the upper plate starts suddenly moving with respect to the lower plate, the normal component starts moving under the action of the viscous force and non-slip condition, whereas the superfluid component will remain initially insensitive to the motion of the plate. Thus, a relative velocity (the counterflow velocity) $\mathbf{V} = \mathbf{v}_n - \mathbf{v}_s$ will arise between the two components. This counterflow velocity $\mathbf{V}$ depends on the distance from the lower plate, in particular $\mathbf{V}$ is maximum for $z = D$ (upper plate) and zero for $z = 0$ (lower plate).

When the counterflow velocity reaches a critical value near the moving plane, the remnant vortices, always present in He II, begin to lengthen and reconnect to form other vortices, which diffuse towards the lower plate (at rest) forming, in the stationary situation, an array of vortices parallel to each other and orthogonal to the flow. Through the vortices, the normal and the superfluid components become coupled by the mutual friction force $\mathbf{F}_{ns}$, and the superfluid will tend to match its velocity with that of the normal fluid ($V \to 0$); this will introduce a rot $\mathbf{v}_s \neq 0$ in the superfluid, expressed by $|\partial \mathbf{v}_s/\partial z|$. After a sufficiently long time, it is expected that a steady shear flow will have formed, with $\mathbf{v}_n = \mathbf{v}_s$ depending only on z and having the x direction and such that $\partial \mathbf{v}_n/\partial z = \partial \mathbf{v}_s/\partial z = \mathbf{V}_0/D$, corresponding to the physical Newtonian linear profile, which follows from (2.8) and (2.9) with vanishing tension force $T = 0$, and (4.12) in which $\mathbf{F}_{ns} = \mathbf{0}$ for $\mathbf{V} = \mathbf{0}$ and $|\mathbf{p}| = 1$. Then, it results $|\text{rot } \mathbf{v}_s| = \mathbf{V}_0/D$.

Introduction of this value in (2.2) would give the areal density of parallel and straight vortex lines, perpendicular to the flow. However, as it has been said in Section 2, the replacement of Ω in terms of rot $\mathbf{v}_s$ is deeper than a formal substitution because $\mathbf{v}_s$ will not become related to the gradient of the barycentric velocity until a complex transient process has lapsed. Thus, the direct replacement of 2Ω in (2.1) by dv_{sx}/dz in shear flows, with v_{sx} the x-component of the macroscopic superfluid velocity, will be valid for steady states and for relatively slow

variations with respect to steady states. Otherwise, rot $\mathbf{v}_s$ with its own nontrivial dynamics should be considered in (2.2). The situation of Couette flow may be rather illustrative of these features.

Then, the dynamics of L in this case is described by

$$\frac{dL}{dt} = -\beta\kappa L^2 + \left[\alpha_1 V + \beta_2 \sqrt{\frac{\kappa}{2}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|}\right] L^{3/2} - \left[\frac{\beta_1}{2}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right| + \beta_4 V \sqrt{\frac{1}{2\kappa}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|}\right] L - \nabla\cdot\mathbf{J}^L, \quad (5.1)$$

where the coefficients should obey the relations indicated below (2.1), and where the last term stands for the effects of the vortex flux for inhomogeneous systems.

In the stationary situation $V \approx 0$ and, according to (5.1), there will be a completely polarized array of vortices, perpendicular to the velocity and to the velocity gradient, with uniform areal density given by

$$L = \frac{1}{\kappa}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right| = \frac{V_0}{\kappa D}. \quad (5.2)$$

In this view, the stationary character of L would require V to be zero, in such a way that normal fluid, superfluid and vortices would move at the same speed and without internal friction. However, equation (5.1) has the intrinsic feature that for V less than a value V_c the vortex line density does not depend on V and is given by (5.2). This critical relative velocity is, according to (5.1),

$$V_c = \frac{\beta}{\alpha_1}\left[2\frac{\beta_4}{\alpha_1} - \frac{\beta_2}{\beta}\right]\sqrt{\frac{\kappa}{2}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|} \cong c'\sqrt{\frac{\kappa}{2}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|}, \quad (5.3)$$

with $c' \approx 3.7$, according to the values of the coefficients used in (2.1) to describe the value of V_c in rotating counterflow velocity.

This indicates that the ordered array of vortices formed in the Couette flow is stable until $V < V_c$. This means that, as V_0 grows, the regular array of rectilinear vortices, orthogonal to V_0, is still present and the velocities $\mathbf{v}_n$, $\mathbf{v}_s$ and $\mathbf{V}$ have rectilinear profiles, but with slightly different slope. The value of V is maximum near the plane $z = D$. When the counterflow velocity V reaches the critical value V_c the regular Couette array of vortices becomes unstable and a disordered tangle of vortex lines appears between the two plates in the zone. If one did not apply (2.1), but only intuitive reasoning without the detailed quantitative analysis showing this critical velocity, one would expect that for $V > 0$ will always be a disordered tangle of vortices.

5.2 Plane Poiseuille flow

Equation (5.1) may be applied to plane Poiseuille flow between two quiescent parallel walls at $z = \pm D/2$, driven by a longitudinal pressure gradient. The steady velocity profile for a Newtonian viscous fluid is parabolic, and has the form

$$V_x(z) = \frac{\triangle p}{2\eta l}\left[\frac{D^2}{4} - z^2\right] = \frac{\triangle p}{\eta l}\frac{D^2}{8}\left[1 - \frac{4z^2}{D^2}\right] = V_{max}\left[1 - \frac{4z^2}{D^2}\right], \quad (5.4)$$

with $\frac{\triangle p}{l}$ the pressure gradient, η the viscosity and $V_{max} = (D^2\triangle p)/(8\eta l)$. The pressure gradient acts on each component in the proportion established by the HVBK equations (2.8)–(2.9).

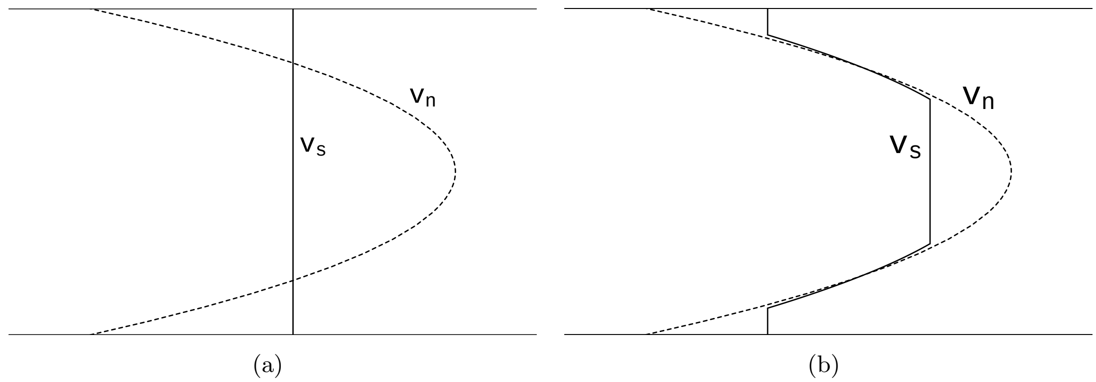

Figure 2: Initial profile (a) and steady profile (b) of the superfluid (continuous line) and normal velocities (dashed line).

Initially, the velocity profile of the normal component, submitted to viscous effects and to no-slip conditions on the walls, will be rather different from that of the superfluid component, which may slip freely along the walls (see Figure 2a). As a useful simplification, one may approximate the velocity profiles as a parabolic (Poiseuille profile) and a flat profile, respectively [19], which equals one to each other in two points at the distance z_0 from the center of the plates. Then, one must search how these profiles will evolve under their mutual interaction due to the friction force, caused by the presence of the vortices.

During the transient regime, vortices will be produced mainly in the regions where the relative velocity V is higher than a critical value V_c — which may be also the central region — but they will be transferred to the matching region where $v_n = v_s$ because of the second term in expression (2.10) for the mutual friction force, which is a Magnus force yielding a vortex lateral drift velocity described by $\mathbf{v}_{L(lateraldrift)} = \alpha' \mathbf{s}' \times \mathbf{V}$. The accumulation of vortices in the region where the two fluids have the same velocity will enlarge the width of the matching region (the profile of $\mathbf{v}_s$ tends to the profile of $\mathbf{v}_n$), until arriving at a situation where V will be lower than V_c so that not more vortices will be produced. The steady profile will have the approximate form of Figure 2b, similar to that considered by Samuels (Fig. 7 of [20]), but in the matching region $\mathbf{v}_n$ and $\mathbf{v}_s$ are not exactly equal, in contrast with Couette flow or rotating cylinder, because there is need of a friction force to cancel out the term in the pressure gradient in the HVBK equations, as shown in (5.6) below.

In the steady state, for isothermal flow, and neglecting the tension $\mathbf{T}$, which vanishes for rectilinear vortices and for isotropic tangles, equations (2.8)–(2.9) reduce to

$$-\frac{\rho_n}{\rho} \nabla p_n + \mathbf{F}_{ns} + \eta \nabla^2 \mathbf{v}_n = 0, \tag{5.5}$$

$$-\frac{\rho_s}{\rho} \nabla p_s - \mathbf{F}_{ns} = 0. \tag{5.6}$$

By adding these equations one obtains $-\nabla p + \eta \nabla^2 \mathbf{v}_n = 0$, which shows that the velocity profile of the normal component is the usual one corresponding to the motion it would have by itself, without the interaction with the superfluid unless some contributions with $\mathbf{T} \neq \mathbf{0}$ would appear, in the form, for instance, of local anisotropy vortex tangles. On the other side, from (5.6) it is seen that $\mathbf{F}_{ns}$ will be different from zero, given by $\mathbf{F}_{ns} = -\frac{\rho_s}{\rho} \nabla p_s$. Thus, $\mathbf{v}_n$ and $\mathbf{v}_s$ will be slightly different, if ∇p is low enough, and there will be an array of straight vortices, which we calculate below.

The most relevant features of the steady profile are: the width $2z_c$ of the central zone without vortices and flat $\mathbf{v}_s$ profile, the width z_w of the boundary layer also without vortices and flat $\mathbf{v}_s$ profile, and $\overline{L}$, the averaged vortex density in the matching zone where vortices concentrate. We will compute them from simple qualitative arguments.

To compute z_c and z_w we will ask that the corresponding circulation of V_{ns} in these regions is lower than the vorticity quantum κ. This is a sufficient condition for the lack of vortices in this zone. The argument is similar to that which could be used to estimate the critical angular velocity for the formation of the first vortex line in a rotating cylinder. If the cylinder is rotating with angular speed Ω, the circulation of V will be $2\pi R^2\Omega$; to obtain Ω_c one equates this quantity to κ, and one gets $\Omega_c = \kappa/(2\pi R^2)$. The exact result is $\Omega_c = \kappa \ln(b/a_0)/(2\pi R^2)$ [1], with a_0 the radius of the vortex line and b a distance of the order of the line spacing, which in the case of the one vortex is of the order of the radius R of the cylinder. In the plane Poiseuille flow situation the quantity b is of the order z_c, in the central zone, and of the order z_w, in the boundary layer zone.

Thus, to estimate z_c we calculate the circulation of $V_{ns} = \frac{\triangle p}{2\eta l}\left[z_c^2 - z^2\right]$ in the zone between $z = 0$ and $z = z_c$ and equate it to $\kappa \ln(b/a_0)$. One has

$$\Gamma_c = \oint_\gamma V_{ns} \cdot \mathrm{d}l = -\int_0^{z_c} \left(\frac{\triangle p}{2\eta l}\left[z_c^2 - z^2\right]\right)\bigg|_{z=0} \mathrm{d}x = \frac{\triangle p}{2\eta l} z_c^3 \approx \kappa \ln(c z_c/a_0) \tag{5.7}$$

where γ is the contour of the square whose side is z_c and c is a numerical constant of the order of the unity. This may be expressed in terms of the maximum velocity V_{max} of $\mathbf{v}_n$ as given by (5.4), leading to expression

$$\frac{z_c^3}{D^3} = \frac{\kappa \ln(c z_c/a_0)}{4 D V_{max}}. \tag{5.8}$$

Concerning the width of the boundary layer z_w, a similar argument yields

$$\Gamma_w = \oint_{\gamma_1} V_{ns} \cdot \mathrm{d}l = \int_0^{z_w} \left(\frac{\triangle p}{2\eta l}\left[\left(\frac{D}{2} - z_w\right)^2 - z^2\right]\right)\bigg|_{z=\frac{D}{2}} \mathrm{d}x = \frac{\triangle p}{2\eta l}\left[D z_w^2 - z_w^3\right] \approx \kappa \ln(c' z_w/a_0), \tag{5.9}$$

where γ_1 is the contour of the square whose side is z_w and c' is a numerical constant of the order of the unity. Up to second order in z_w, this yields

$$\frac{\triangle p}{2\eta l} D z_w^2 = \kappa \ln(c' z_w/a_0), \tag{5.10}$$

and using expression (5.4) for the $\mathbf{v}_n$ profile, the previous expression may be rewritten in terms of V_{max} as

$$\frac{z_w^2}{D^2} = \frac{\kappa \ln(c' z_w/a_0)}{4 D V_{max}}. \tag{5.11}$$

This expression is similar to the one obtained by Samuels in [20] for the width of the outer layer in a cylindrical Poiseuille flow in a tube of diameter D (his eq. (15)), which was

$$\left(\frac{r_c}{D}\right)^2 = \frac{\kappa}{8\pi D V_{max}} \ln\left(\frac{8 r_c}{a_0}\right). \tag{5.12}$$

From (5.8) and (5.11) it is found that the widths z_c and z_w decrease for increasing V_{max} as $z_c \sim V_{max}^{-1/3}$ and $z_w \sim V_{max}^{-1/2}$. Thus for increasing V_{max} (i.e. increasing pressure gradient) the central zone and the outer zone boundary layer free of vortices will become thinner. The flat

profile of $\mathbf{v}_s$ in these zones is consistent with the absence of vortices, according to the relation $L = |\partial \mathbf{v}_s/\partial z|/\kappa$, analogous to the expression (5.2), and which vanishes for flat profile.

In the matching region the value of $\mathbf{v}_n - \mathbf{v}_s$ will not be strictly zero, but because of restriction (5.6) if $\mathbf{v}_n - \mathbf{v}_s$ is approximately constant in this region, one will have that

$$L(z) = \frac{1}{\kappa}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right| \approx \frac{1}{\kappa}\left|\frac{\partial \mathbf{v}_n}{\partial z}\right| = \frac{8V_{max}}{\kappa D^2}|z|. \tag{5.13}$$

It is known that there exist two values of z where the velocities, $\mathbf{v}_n$ and $\mathbf{v}_s$, are equal, but, in general, in the rest of the z domain they could not be exactly equal. This implies that the mutual friction force could depend on z and that the distribution of the vortices could not be homogeneous. To overcome this problem, we average the value of L in the region between $z = z_c$ and $z = z_1 = D/2 - z_w$. Of course, the value of $\overline{L}$ in the region between $z = -z_1 = -D/2 + z_w$ and $z = -z_c$ will be the same of the first region by symmetry. To estimate, we assume that the averaged profile of the superfluid velocity can be approximated by the profile of the normal velocity, so obtaining

$$\overline{L} = \frac{8V_{max}}{\kappa D^2}\left|\frac{z_c - z_1}{2}\right| = \frac{2V_{max}}{\kappa D}\left|1 + 2\left(\frac{z_c}{D} - \frac{z_w}{D}\right)\right|. \tag{5.14}$$

Introducing z_c and z_w as obtained from (5.8) and (5.11) we would have an estimate of $\overline{L}$ in terms of V_{max}, or, equivalently, in terms of $\triangle p$. A more detailed analysis could be carried out from the transversal terms of the vortex flux, where the Magnus drift and the diffusion flux in (2.7) would cancel each other.

Expression (5.8) may be used to obtain the conditions for a laminar flow without any vortex. This situation will be found when the width of the central zone without vortices z_c is bigger than $D/2$. This leads to the condition $DV_{max}/\kappa \leq 2\ln(D/(2a_0))$. For $D \approx 10^{-2}$m, and since $a_0 \approx 10^{-10}$m, we have $DV_{max}/\kappa \leq 20$. The dimensionless quantity DV_{max}/κ is analogous to the Reynolds number. In viscous fluid, the Reynolds number is defined as DV/ν, with ν being the kinematic viscosity $\nu = \eta/\rho$, which has dimensions m^{2}s^{-1}. The vorticity quantum κ has also dimension m^{2}s^{-1} and therefore DV_{max}/κ may be considered as a quantum Reynolds number. A similar number has been used in pure counterflow experiments in cylindrical containers of diameter D where, for instance, the appearance of the first vortex takes place at $T = 1.7$K for $DV/\kappa \approx 80$ [21]. A more rigorous stability analysis would be convenient to obtain more values of the critical quantum Reynolds number in both situations.

6 Conclusions

The quantized character of vorticity in superfluids emphasizes the special importance of vortex lines, whose dynamics becomes a central aspect of rotating or turbulent flows of superfluids. The main proposal of this paper is equation (2.2) for the evolution of vortex line density, which generalizes our previous proposal (2.1) for rotating counterflow situations. Here, by writing the local average rotational of the superfluid component instead of the angular velocity, we have enlarged the set of applications of the theory in two main aspects. One of them is that (2.2), in contrast to (2.1), may be applied not only to rotation but also to shear flows, as illustrated in Section 5. The second enlargement is of dynamical nature: in (2.2) rot $\mathbf{v}_s$ appears, and $\mathbf{v}_s$ itself should satisfy its own evolution equation, which is coupled to the evolution of $\mathbf{v}_n$, the velocity of the normal component. Then, (2.2) becomes deeply coupled to the HVBK

equations (2.8) and (2.9) for $\mathbf{v}_n$ and $\mathbf{v}_s$ not only through the mutual force $\mathbf{F}_{ns}$ between the normal and the superfluid components, which requires the knowledge of L, but also because in (2.2) $\mathbf{v}_s$ is needed to obtain L. Thus, the coupling of these equations is much emphasized in (2.2) as compared to (2.1).

For situations close to nonequilibrium steady states or for slow variations of $\mathbf{v}_s$, in such a way that rot $\mathbf{v}_s$ is well described by the angular velocity or by the barycentric velocity gradient, equations (2.1) or (5.1) describe the vortex line density in terms of Ω or dv_{sx}/dz. In each case we have provided an estimation of the vortex density and of the superfluid velocity profile in the steady state.

We have compared our proposal with that of Lipniacki, which shares the objectives of the present paper but stresses the polarization $\mathbf{p} = \text{rot } \mathbf{v}_s/kL$ more than rot $\mathbf{v}_s$ itself. Lipniacki's evolution equation for L is, essentially, the classical Vinen's equation, but with the new aspect that its coefficients become dependent on the polarization $\mathbf{p}$ according to the microscopic identification of the coefficients proposed by Schwarz [10]. The disagreement between our equation (2.5) and the Lipniacki's proposal (3.10) may be due to the different physical origin of the terms dependent on the polarization. Our opinion is that Schwarz derivation of Vinen's equation (3.5) does not include some relevant contributions of rotational systems. For straight parallel vortices, as those arising in pure rotation experiments, both the production and the destruction terms vanish. This is consistent with Schwarz's postulates for the vortices, but in purely rotational flows the dynamics of vortices has a different origin, related to the migration of vortices formed on the wall towards the center of the system, and with repulsion forces amongst vortices. Thus a general treatment would require to include these effects besides the Scharwz effects in (3.5), and it could provide a further understanding of the differences between (2.5) and (3.10). In any case, comparison with experimental results in Fig. 1 indicates that (2.5) yields a better description of them.

Acknowledgments

We acknowledge the support of the Acción Integrada España-Italia (Grant S2800082F HI2004-0316 of the Spanish Ministry of Science and Technology and grant IT2253 of the Italian MIUR). DJ acknowledges the financial support from the Dirección General de Investigación of the Spanish Ministry of Education under grant FIS 2006-12296-C02-01 and of the Direcció General de Recerca of the Generalitat of Catalonia, under grant 2005 SGR-00087. MSM and MS acknowledge the financial support from MIUR under grant "PRIN 2005 17439-003" and by "Fondi 60%" of the University of Palermo. MS acknowledges the "Assegno di ricerca: Studio della turbolenza superfluida e della sua evoluzione" of the University of Palermo.

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